ON THE DEFINING IDEAL OF A SET OF POINTS IN MULTI-PROJECTIVE SPACE

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Abstract. We investigate the defining ideal $I_{\mathbb{X}}$ of a set of points $\mathbb{X}$ in $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$ with a special emphasis on the case that $\mathbb{X}$ is in generic position, that is, $\mathbb{X}$ has the maximal Hilbert function. When $\mathbb{X}$ is in generic position, we determine the degrees of the generators of the associated ideal $I_{\mathbb{X}}$. Letting $\nu(I_{\mathbb{X}})$ denote the minimal number of generators of $I_{\mathbb{X}}$, we use this description of the degrees to construct a function $v(s; n_1, \ldots, n_k)$ with the property that $\nu(I_{\mathbb{X}}) \geq v(s; n_1, \ldots, n_k)$ always holds for s points in generic position in $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$. When $k = 1$, $v(s; n_1)$ equals the expected value for $\nu(I_{\mathbb{X}})$ as predicted by the Ideal Generation Conjecture. If $k \geq 2$, we show that there are cases with $\nu(I_{\mathbb{X}}) > v(s; n_1, \ldots, n_k)$. However, computational evidence suggests that in many cases $\nu(I_{\mathbb{X}}) = v(s; n_1, \ldots, n_k)$.

Introduction

In this paper we investigate the generators of the ideal $I_{\mathbb{X}}$ defining a set of points $\mathbb{X}$ in generic position in $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$.

One of the fundamental open problems about finite sets of points $\mathbb{X} \subseteq \mathbb{P}^n$ in generic position, i.e., those sets of points having the maximal Hilbert function, is to count the minimal number of generators of $I_{\mathbb{X}}$ in terms of the data n and $|\mathbb{X}| = s$. This question is the content of the Ideal Generation Conjecture (IGC) (see [7]). Recently, many authors (cf. [2, 8, 9, 10, 11, 12, 16, 17]) have been interested in generalizing results about points in $\mathbb{P}^n$ to $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$. We continue this program by studying the generators of $I_{\mathbb{X}}$ when $\mathbb{X} \subseteq \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$ with the hope that this might lead to a generalized IGC. Our investigation was also partially motivated by the desire to understand which properties about the ideal of points in $\mathbb{P}^n$, specifically those shown in [4, 6, 7], carry over to $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$.

Given the defining ideal $I_{\mathbb{X}}$ of a set of points $\mathbb{X} \subseteq \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$, two natural questions about the generators of $I_{\mathbb{X}}$ arise: (1) what are the degrees of the generators? and (2) what is $\nu(I_{\mathbb{X}}) :=$ minimal number of generators of $I_{\mathbb{X}}$? These questions can be viewed as the first step in describing the multi-graded minimal free resolution of $I_{\mathbb{X}}$ since (1) and (2) are questions about the 0th multi-graded Betti numbers.

In Section 2 we show that the Hilbert function of a set of points can be used to bound the degrees of the generators, thus giving a partial answer to (1). As posed, however, these questions are difficult to attack, even when $k = 1$, without further conditions on the points.

For finite sets of points $\mathbb{X} \subseteq \mathbb{P}^n$, these questions have been primarily studied under the extra hypothesis that the set of points is in generic position, i.e., $H_{\mathbb{X}}(i) = \min\{\dim_{\mathbf{k}} R_i, |\mathbb{X}|\}$ for all $i \in \mathbb{N}$. Thus, one is led to ask about the generators of $I_{\mathbb{X}}$ when $\mathbb{X}$ is a set of points in generic position in $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$. However, it is first necessary to establish the basic properties (like existence) of points in generic position in multi-projective spaces since these facts are not part of the literature. Analogous to the case of points in $\mathbb{P}^n$, we say that a set $\mathbb{X}$ of s points in $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$ is in *generic position* if $H_{\mathbb{X}}(\underline{i}) = \min\{\dim_{\mathbf{k}} R_{\underline{i}}, s\}$ for all $\underline{i} \in \mathbb{N}^k$. In Section 3 we show that these points exist, and moreover, if we consider each set of s points as a point in $(\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k})^s$, the points in generic position form a non-empty open subset of $(\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k})^s$ with respect to the Zariski topology.

2000 *Mathematics Subject Classification.* 13D02,13D40, 14A05.
Key words and phrases. points, multi-projective space, generators.
Supported in part by the Natural Sciences and Engineering Research Council of Canada.

We also show in Section 3 that if F is a generator of $I_{\mathbb{X}}$, then $\deg F = \underline{i}$ or $\underline{i} + e_j$ where $\underline{i} \in \mathcal{D} := \min\left\{\underline{i} \in \mathbb{N}^k \;\middle|\; \binom{i_1+n_1}{i_1}\cdots\binom{i_k+n_k}{i_k} > s\right\}$ and e_j is one of the k basis vectors of $\mathbb{N}^k$. This result gives an answer to (1) and generalizes the fact that $I_{\mathbb{X}} = \langle (I_{\mathbb{X}})_d \oplus (I_{\mathbb{X}})_{d+1}\rangle$ with $d = \min\left\{i \;\middle|\; \binom{i+n}{i} > s\right\}$ in the graded case. An interesting difference between points in generic position in $\mathbb{P}^n$ versus $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$ is that $R/I_{\mathbb{X}}$ is *always* Cohen-Macaulay if $k = 1$, but is *never* Cohen-Macaulay if $k \geq 2$ (see Theorem 3.4)

In Section 4 we use this description of the degrees to show that $\nu(I_{\mathbb{X}})$ can be determined by counting the generators of degree $\underline{i}$ and $\underline{i} + e_j$ for all $\underline{i} \in \mathcal{D}$ and $j = 1, \ldots, k$. By degree considerations, $I_{\mathbb{X}}$ has $\dim_{\mathbf{k}}(I_{\mathbb{X}})_{\underline{i}}$ generators of degree $\underline{i}$ for each $\underline{i} \in \mathcal{D}$. To count the generators of degree $\underline{i} + e_j$, we need to calculate the dimension of the image of the map $\Phi_{\underline{i},j} : R_{e_j} \otimes_{\mathbf{k}} (I_{\mathbb{X}})_{\underline{i}} \xrightarrow{a\times b} (I_{\mathbb{X}})_{\underline{i}+e_j}$ for each $\underline{i} \in \mathcal{D}$ and $1 \leq j \leq k$. Moreover, if there exists $\underline{i}_1, \underline{i}_2 \in \mathcal{D}$ and $1 \leq j_1, j_2 \leq k$ such that $l = \underline{i}_1 + e_{j_1} = \underline{i}_2 + e_{j_2}$, then $\operatorname{Im} \Phi_{\underline{i}_1, e_{j_1}}$ and $\operatorname{Im} \Phi_{\underline{i}_1, e_{j_2}}$ are both subspaces of $(I_{\mathbb{X}})_l$, so we also need to know $\dim_{\mathbf{k}}(\operatorname{Im}_{\underline{i}_1, e_{j_1}} \cap \operatorname{Im} \Phi_{\underline{i}_1, e_{j_2}})$. In general, it is difficult to compute the sizes of these vector spaces, even if $k = 1$, except in some special cases. When $k = 1$, to compute $\nu(I_{\mathbb{X}})$ only the dimension of the image of $\Phi : R_1 \otimes_{\mathbf{k}} (I_{\mathbb{X}})_d \to (I_{\mathbb{X}})_{d+1}$ needs to be calculated. The IGC states that $\operatorname{Im} \Phi$ should be as large as possible for a sufficiently general set of points (a subset of those points in generic position).

By considering the largest possible value for each $\dim_{\mathbf{k}} \operatorname{Im} \Phi_{\underline{i},j}$, in Section 5 we construct a function $v(s; n_1, \ldots, n_k)$ with the property that $\nu(I_{\mathbb{X}}) \geq v(s; n_1, \ldots, n_k)$ always holds for a set of s points in generic position in $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$. When $k = 1$, $v(s; n)$ equals the expected value for $\nu(I_{\mathbb{X}})$ as predicted by the IGC. Furthermore, using [9, 10], we show that $\nu(I_{\mathbb{X}}) = v(s; 1, 1)$ for a sufficiently general set of s points in $\mathbb{P}^1 \times \mathbb{P}^1$.

Buoyed by these results, we had hoped that for any set of s points in generic position that were sufficiently general, we should expect $\nu(I_{\mathbb{X}}) = v(s; n_1, \ldots, n_k)$. However, we show that if $\mathbb{X}$ is any three points in generic position in $\mathbb{P}^1 \times \cdots \times \mathbb{P}^1$ ($k \geq 3$ times), then $\nu(I_{\mathbb{X}}) > v(3; 1, \ldots, 1)$. As well, computational evidence suggests that $\nu(I_{\mathbb{X}}) > v(s; 1, n, n)$ if $s = 1 + n + n$. These cases appear to be exceptional because in all other computed examples the equality $\nu(I_{\mathbb{X}}) = v(s; n_1, \ldots, n_k)$ holds. Moreover, we know of no example of $\nu(I_{\mathbb{X}}) > v(s; n_1, n_2)$ when $k = 2$. This leads us to believe that $\nu(I_{\mathbb{X}}) = v(s; n_1, \ldots, n_k)$ in a large number of cases, thus giving us a partial generalization of the the IGC.

1. Preliminaries

In this paper $\mathbf{k}$ denotes a field with $\operatorname{char}(\mathbf{k}) = 0$ and $\mathbf{k} = \overline{\mathbf{k}}$. This section provides the relevant facts and definitions about multi-graded rings, Hilbert functions, and sets of points in multi-projective spaces. See also [15, 16, 17].

Let $\mathbb{N} := \{0, 1, 2, \ldots\}$. For any integer $k \geq 1$, we write $[k] := \{1, \ldots, k\}$. We denote $(i_1, \ldots, i_k) \in \mathbb{N}^k$ by $\underline{i}$. We set $|\underline{i}| := \sum_h i_h$. If $\underline{i}, \underline{j} \in \mathbb{N}^k$, then $\underline{i} + \underline{j} := (i_1 + j_1, \ldots, i_k + j_k)$. We write $\underline{i} \geq \underline{j}$ if $i_h \geq j_h$ for every $h = 1, \ldots, k$. Observe that $\geq$ is a partial order on $\mathbb{N}^k$. For any subset $\mathcal{A} \subseteq \mathbb{N}^k$, we will use $\min \mathcal{A}$ to denote the set of minimal elements of $\mathcal{A}$ with respect to this partial order. The set $\mathbb{N}^k$ is a semi-group generated by $\{e_1, \ldots, e_k\}$ where $e_i := (0, \ldots, 1 \ldots, 0)$ is the ith standard basis vector of $\mathbb{N}^k$. For any $c \in \mathbb{N}$, $ce_i := (0, \ldots, c, \ldots, 0)$.

Set $R = \mathbf{k}[x_{1,0}, \ldots, x_{1,n_1}, x_{2,0}, \ldots, x_{2,n_2}, \ldots, x_{k,0}, \ldots, x_{k,n_k}]$, and induce an $\mathbb{N}^k$-grading on R by setting $\deg x_{i,j} = e_i$. An element $x \in R$ is said to be $\mathbb{N}^k$-*homogeneous* (or simply *homogeneous* if the grading is clear) if $x \in R_{\underline{i}}$ for some $\underline{i} \in \mathbb{N}^k$. If x is homogeneous, then $\deg x := \underline{i}$.

An ideal $I = (F_1, \ldots, F_r) \subseteq R$ is an $\mathbb{N}^k$-*homogeneous* (or, simply *homogeneous*) *ideal* if each F_j is $\mathbb{N}^k$-homogeneous. If $I \subseteq R$ is a homogeneous ideal, $S = R/I$ inherits an $\mathbb{N}^k$-graded ring structure if we define $S_{\underline{i}} = (R/I)_{\underline{i}} := R_{\underline{i}}/I_{\underline{i}}$. The function $H_S(\underline{i}) := \dim_{\mathbf{k}}(R/I)_{\underline{i}}$ is the *Hilbert function of* S.

If I is an $\mathbb{N}^k$-homogeneous ideal of R, then for any $\underline{i} \in \mathbb{N}^k$, and for any $j \in [k]$, we set

$$R_{e_j} I_{\underline{i}} := \left\{ f \;\middle|\; f = f_0 x_{j,0} + f_1 x_{j,1} + \cdots + f_{n_j} x_{j,n_j},\ f_l \in I_{\underline{i}} \right\}.$$

Note that $R_{e_j} I_{\underline{i}}$ is a subspace of the vector space $I_{\underline{i}+e_j}$.

For every $\underline{i} \in \mathbb{N}^k$, a basis for $R_{\underline{i}}$ as a vector space over $\mathbf{k}$ is the set of all monomials in R of degree $\underline{i}$. Thus, $\dim_{\mathbf{k}} R_{\underline{i}} = \binom{n_1+i_1}{i_1}\binom{n_2+i_2}{i_2}\cdots\binom{n_k+i_k}{i_k}$. We set $N(\underline{i}) := \dim_{\mathbf{k}} R_{\underline{i}}$ for each $\underline{i} \in \mathbb{N}^k$.

The $\mathbb{N}^k$-graded ring R is the coordinate ring of $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$. If $P \in \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$ is a point, and if I_P denotes the ideal associated to P, then the ideal I_P is a prime ideal, and furthermore, $I_P = (L_{1,1}, \ldots, L_{1,n_1}, \ldots, L_{k,1}, \ldots, L_{k,n_k})$ where $\deg L_{i,j} = e_i$ for $j = 1, \ldots, n_i$. Let $P_1, \ldots, P_s$ be s distinct points in $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$. If $\mathbb{X} = \{P_1, \ldots, P_s\}$, then the $\mathbb{N}^k$-homogeneous ideal $I_{\mathbb{X}}$ of forms that vanish at $\mathbb{X}$ is $I_{\mathbb{X}} = I_{P_1} \cap \cdots \cap I_{P_s}$ where I_{P_i} is the ideal associated to the point P_i. The coordinate ring $R/I_{\mathbb{X}}$ then has the following property.

Lemma 1.1 ([16, Lemma 3.3]). *If $\mathbb{X} \subseteq \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$ is a finite set of points, then for each $l \in [k]$, there exists $L_l \in R_{e_l}$ such that $\overline{L}_l$ is a non-zero divisor in $R/I_{\mathbb{X}}$.*

Remark 1.2. After a linear change of variables in the $x_{1,j}$'s, a change of variables in the $x_{2,j}$'s, and so on, we can take $L_l = x_{l,0}$ for each $l \in [k]$. We therefore assume, once and for all, that the set of points $\mathbb{X}$ under investigation has the property that $\overline{x}_{l,0}$ is a non-zero divisor in $R/I_{\mathbb{X}}$ for each $l \in [k]$.

We sometimes write $H_{\mathbb{X}}$ for $H_{R/I_{\mathbb{X}}}$, and call $H_{\mathbb{X}}$ the Hilbert function of $\mathbb{X}$. Classifying the Hilbert functions of sets of points in $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$ with $k \geq 2$ remains an open problem (the case $k = 1$ is dealt with in [5]). See [8] and [16] for some progress on this problem. However, some growth conditions on $H_{\mathbb{X}}$ are known.

Theorem 1.3 ([16, Proposition 3.5]). *Let $\mathbb{X}$ be a finite set of points in $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$ with Hilbert function $H_{\mathbb{X}}$.*

(i) *For all $\underline{i} \in \mathbb{N}^k$, $H_{\mathbb{X}}(\underline{i}) \leq H_{\mathbb{X}}(\underline{i} + e_l)$ for all $l \in [k]$.*
(ii) *If $H_{\mathbb{X}}(\underline{i}) = H_{\mathbb{X}}(\underline{i} + e_l)$ for some $l \in [k]$, then $H_{\mathbb{X}}(\underline{i} + e_l) = H_{\mathbb{X}}(\underline{i} + 2e_l)$.*

Let $\pi_i : \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k} \to \mathbb{P}^{n_i}$ denote the ith projection morphism defined by $P_1 \times \cdots \times P_i \times \cdots \times P_k \mapsto P_i$. Then $\pi_i(\mathbb{X})$ is the set of all the ith-coordinates in $\mathbb{X}$. For each $i \in [k]$, set $t_i := |\pi_i(\mathbb{X})|$. With this notation we have

Theorem 1.4 ([16, Corollary 4.7]). *Let $\mathbb{X}$ be a finite set of points in $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$ with Hilbert function $H_{\mathbb{X}}$. Fix an $i \in [k]$. If $(j_1, \ldots, j_i, \ldots, j_k) \in \mathbb{N}^k$ with $j_i \geq t_i - 1$, then $H_{\mathbb{X}}(j_1, \ldots, j_i, \ldots, j_k) = H_{\mathbb{X}}(j_1, \ldots, t_i - 1, \ldots, j_k)$.*

Remark 1.5. One can interpret the above results as follows. Fix an integer $i \in [k]$, and fix $k-1$ integers in $\mathbb{N}$, say $j_1, \ldots, j_{i-1}, j_{i+1}, \ldots, j_k$. Set

$$\underline{j}_l := (j_1, \ldots, j_{i-1}, l, j_{i+1}, \ldots, j_k) \text{ for each integer } l \in \mathbb{N}.$$

Then Theorems 1.3 and 1.4 imply that there exists an integer $l' \leq t_i - 1$ such that the sequence $H_{\mathbb{X}}(\underline{j}_0), H_{\mathbb{X}}(\underline{j}_1), H_{\mathbb{X}}(\underline{j}_2), H_{\mathbb{X}}(\underline{j}_3), \ldots.$ has the property that $H_{\mathbb{X}}(\underline{j}_l) < H_{\mathbb{X}}(\underline{j}_{l+1})$ if $0 \leq l < l'$, but $H_{\mathbb{X}}(\underline{j}_l) = H_{\mathbb{X}}(\underline{j}_{l+1})$ if $l \geq l'$.

2. On the generators of an ideal of a set of points

Let $I_{\mathbb{X}}$ be the defining ideal of a finite set of points $\mathbb{X} \subseteq \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$ with Hilbert function $H_{\mathbb{X}}$. Using only $H_{\mathbb{X}}$, we describe a finite subset $\mathcal{E} \subseteq \mathbb{N}^k$ with the property that if F is a generator of $I_{\mathbb{X}}$, then $\deg F \in \mathcal{E}$.

Fix an $l \in [k]$. Then, for each $\underline{j} = (j_1, \ldots, j_{l-1}, j_{l+1}, \ldots, j_k) \in \mathbb{N}^{k-1}$, set

$$i(\underline{j}) := \min\left\{ i \in \mathbb{N}^+ \;\middle|\; \begin{array}{c} H_{\mathbb{X}}(j_1, \ldots, j_{l-1}, i-1, j_{l+1}, \ldots, j_k) = \\ H_{\mathbb{X}}(j_1, \ldots, j_{l-1}, i, j_{l+1}, \ldots, j_k) \end{array} \right\}.$$

The existence of the integer $i(\underline{j})$ follows from Theorem 1.4.

Theorem 2.1. *Let $\mathbb{X}$ be a finite set of points of $\mathbb{P}^{n_1}\times\cdots\times\mathbb{P}^{n_k}$. Fix an $l \in [k]$ and $\underline{j} = (j_1,\ldots,j_{l-1},j_{l+1},\ldots,j_k) \in \mathbb{N}^{k-1}$. Set $\underline{i} = (j_1,\ldots,j_{l-1},i(\underline{j}),j_{l+1},\ldots,j_k)$. Then*

$$(I_{\mathbb{X}})_{\underline{i}+(r+1)e_l} = R_{e_l}(I_{\mathbb{X}})_{\underline{i}+re_l} \quad \text{for all } r \in \mathbb{N}.$$

In particular, if there exists $\underline{l} \in \mathbb{N}^k$ and $t \in [k]$ such that $H_{\mathbb{X}}(\underline{l}) = H_{\mathbb{X}}(\underline{l}-e_t) = H_{\mathbb{X}}(\underline{l}-2e_t)$, then $I_{\mathbb{X}}$ has no minimal generators of degree $\underline{l}$.

Proof. Without loss of generality, we only consider the case $l = 1$. By Remark 1.2 we can take $x_{1,0}$ to be a non-zero divisor. Set $S = \mathbf{k}[x_{1,1},\ldots,x_{k,n_k}] \cong R/(x_{1,0})$ and $\underline{i} = (i(\underline{j}),j_2,\ldots,j_k)$ where $\underline{j} = (j_2,\ldots,j_k)$.

Because $x_{1,0}$ is a non-zero divisor, $I_{\mathbb{X}}/x_{1,0}I_{\mathbb{X}} \cong ((I_{\mathbb{X}},x_{1,0})/x_{1,0})$. For each $\underline{t} \in \mathbb{N}^k$ the short exact sequence

$$0 \longrightarrow (I_{\mathbb{X}})_{\underline{t}-e_1} \xrightarrow{\times x_{1,0}} (I_{\mathbb{X}})_{\underline{t}} \longrightarrow (I_{\mathbb{X}}/(x_{1,0}I_{\mathbb{X}}))_{\underline{t}} \longrightarrow 0$$

implies $\dim_{\mathbf{k}}(I_{\mathbb{X}})_{\underline{t}} = \dim_{\mathbf{k}}(I_{\mathbb{X}})_{\underline{t}-e_1} + \dim_{\mathbf{k}}((I_{\mathbb{X}},x_{1,0})/x_{1,0})_{\underline{t}}$. On the other hand, the short exact sequence

$$0 \longrightarrow R/I_{\mathbb{X}}(-e_1) \xrightarrow{\times \overline{x}_{1,0}} R/I_{\mathbb{X}} \longrightarrow R/(I_{\mathbb{X}},x_{1,0}) \cong \frac{R/(x_{1,0})}{(I_{\mathbb{X}},x_{1,0})/x_{1,0}} \longrightarrow 0,$$

and the hypothesis that $\dim_{\mathbf{k}}(R/I_{\mathbb{X}})_{\underline{i}+re_1} = \dim_{\mathbf{k}}(R/I_{\mathbb{X}})_{\underline{i}+(r-1)e_1}$ for every $r \in \mathbb{N}$ implies that $((I_{\mathbb{X}},x_{1,0})/x_{1,0})_{\underline{i}+re_1} = (R/(x_{1,0}))_{\underline{i}+re_1} \cong S_{\underline{i}+re_1}$ for all $r \in \mathbb{N}$.

Fix an integer $r \in \mathbb{N}$ and set $W = R_{e_1}(I_{\mathbb{X}})_{\underline{i}+re_1}$. Then $W \subseteq (I_{\mathbb{X}})_{\underline{i}+(r+1)e_1}$, and because $x_{1,0}$ is a non-zero divisor

$$\begin{aligned}\dim_{\mathbf{k}} W &= \dim_{\mathbf{k}}(I_{\mathbb{X}})_{\underline{i}+re_1} + \dim_{\mathbf{k}} W' \\ &= \dim_{\mathbf{k}}(I_{\mathbb{X}})_{\underline{i}+(r+1)e_1} - \dim_{\mathbf{k}} S_{\underline{i}+(r+1)e_1} + \dim_{\mathbf{k}} W'\end{aligned}$$

where $W' = \{f'' \mid f = f'x_{1,0} + f'',\ f \in W\} = \{f(0,x_{1,1},\ldots,x_{k,n_k}) \mid f \in W\}$. By slightly abusing notation, W' can be viewed as a subset of $S_{\underline{i}+(r+1)e_1}$.

It suffices to show that $S_{\underline{i}+(r+1)e_1} \subseteq W'$ because then $\dim_{\mathbf{k}} S_{\underline{i}+(r+1)e_1} = \dim_{\mathbf{k}} W'$, whence $\dim_{\mathbf{k}} W = \dim_{\mathbf{k}}(I_{\mathbb{X}})_{\underline{i}+(r+1)e_1}$ which gives $W = (I_{\mathbb{X}})_{\underline{i}+(r+1)e_1}$. So, suppose $f \in S_{\underline{i}+(r+1)e_1}$. Then $f = f_1x_{1,1} + \cdots + f_{n_1}x_{1,n_1}$ with $f_i \in S_{\underline{i}+re_1}$. But $S_{\underline{i}+re_1} \cong ((I_{\mathbb{X}},x_{1,0})/x_{1,0})_{\underline{i}+re_1}$, so by abusing notation, there exists $F_i \in (I_{\mathbb{X}})_{\underline{i}+re_1}$ such that $F_i = g_ix_{1,0} + f_i$. But then $F = F_1x_{1,1} + \cdots + F_{n_1}x_{1,n_1} \in W$, whence $f \in W'$.

For the last statement let $\underline{l} = (l_1,\ldots,l_k)$ and $i := i(l_1,\ldots,l_{t-1},l_{t+1},\ldots,l_k)$. Then $H_{\mathbb{X}}(\underline{l}) = H_{\mathbb{X}}(\underline{l}-e_t) = H_{\mathbb{X}}(\underline{l}-2e_t)$ implies that $\underline{l}-e_t \geq (l_1,\ldots.l_{t-1},i,l_{t+1},\ldots,l_k)$, so $(I_{\mathbb{X}})_{\underline{l}} = R_{e_t}(I_{\mathbb{X}})_{\underline{l}-e_t}$. □

Let $\mathbb{X}$ be a finite set of points in $\mathbb{P}^{n_1}\times\cdots\times\mathbb{P}^{n_k}$, and set

$$\mathcal{B} := \left\{\underline{i} \in \mathbb{N}^k \;\middle|\; H_{\mathbb{X}}(\underline{i}) < \dim_{\mathbf{k}} R_{\underline{i}} = N(\underline{i})\right\} = \left\{\underline{i} \in \mathbb{N}^k \;\middle|\; (I_{\mathbb{X}})_{\underline{i}} \neq 0\right\}.$$

For each $l \in [k]$ and for each $\underline{j} = (j_1,\ldots,j_{l-1},j_{l+1},\ldots,j_k) \in \mathbb{N}^{k-1}$ set

$$\mathcal{A}_{l,\underline{j}} := \left\{(j_1,\ldots,j_{l-1},i(\underline{j}),j_{l+1},\ldots,j_k) + re_l \;\middle|\; r \in \mathbb{N}^+\right\}.$$

We then define

$$\mathcal{A} := \bigcup_{l=1}^{k}\left(\bigcup_{\underline{j}\in\mathbb{N}^{k-1}} \mathcal{A}_{l,\underline{j}}\right).$$

Theorem 2.2. *Let $\mathbb{X}$ be a finite set of points in $\mathbb{P}^{n_1}\times\cdots\times\mathbb{P}^{n_k}$ with defining ideal $I_{\mathbb{X}}$. With the notation as above, set $\mathcal{E} = \mathcal{B}\backslash\mathcal{A}$. Then $\mathcal{E}$ is a finite set. Furthermore, if f is a generator of $I_{\mathbb{X}}$, then $\deg f \in \mathcal{E}$. In particular, $I_{\mathbb{X}} = \left\langle \bigoplus_{\underline{i}\in\mathcal{E}}(I_{\mathbb{X}})_{\underline{i}}\right\rangle$.*

Proof. We show that $\mathcal{E}$ is finite. Let $t_i := |\pi_i(\mathbb{X})|$ where $\pi_i : \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k} \to \mathbb{P}^{n_i}$ is the ith projection morphism. Set $\mathcal{F} := \{\underline{i} \in \mathbb{N}^k \mid \underline{i} \leq (t_1, t_2, \ldots, t_k)\}$. Now suppose that $\underline{j} = (j_1, \ldots, j_k) \in \mathbb{N}^k \backslash \mathcal{F}$. Thus, there is a coordinate of $\underline{j}$, say j_i, such that $j_i \geq t_i + 1$. By Theorem 1.4

$$H_{\mathbb{X}}(\underline{j}) = H_{\mathbb{X}}(j_1, \ldots, j_{i-1}, t_i, j_{i+1}, \ldots, j_k) = H_{\mathbb{X}}(j_1, \ldots, j_{i-1}, t_i - 1, j_{i+1}, \ldots, j_k).$$

This means that $\underline{j} \in \mathcal{A}_{i,(j_1,\ldots,j_{i-1},j_{i+1},\ldots,j_k)} \subseteq \mathcal{A}$. We thus have $\mathbb{N}^k \backslash \mathcal{F} \subseteq \mathcal{A}$. But this implies that $\mathcal{E} = \mathcal{B} \backslash \mathcal{A} \subseteq \mathbb{N}^k \backslash \mathcal{A} \subseteq \mathcal{F}$, and since $\mathcal{F}$ is finite, so is $\mathcal{E}$.

For the second statement, let f be a generator of $I_{\mathbb{X}}$. Then it is immediate that $\deg f \in \mathcal{B}$. On the other hand, Theorem 2.1 implies that $\deg f \notin \mathcal{A}_{l,\underline{j}}$ for any $l \in [k]$ or $\underline{j} \in \mathbb{N}^{k-1}$. Hence, $\deg f \notin \mathcal{A}$, so $\deg f \in \mathcal{E}$. $\square$

Remark 2.3. We recover Proposition 1.1 (3) of [4] when $k = 1$, i.e., $\mathbb{X} \subseteq \mathbb{P}^n$.

3. Points in generic position

Analogous to the definition for points in $\mathbb{P}^n$, a set of points in $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$ is said to be in generic position if its Hilbert function is maximal. Although such sets have been studied (cf. [9, 10]) we could find no proof in the literature for the existence of such sets when $k \geq 2$ (the case $k = 1$ is [6, Theorem 4]). We therefore begin by providing a proof of this "folklore" result. Then, if $I_{\mathbb{X}}$ is the defining ideal of a set of points in generic position in $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$, we compute the depth of $R/I_{\mathbb{X}}$, and give bounds on the degrees of the generators of $I_{\mathbb{X}}$.

Theorems 1.3 and 1.4 imply the number of possible Hilbert functions for s points is finite. However, since the number of sets with s points is infinite, we can ask if there exists an expected Hilbert function for s points. We give a heuristic argument for this expected function.

If $\{m_1, \ldots, m_{N(\underline{j})}\}$ are the $N(\underline{j})$ monomials of degree $\underline{j}$ in the $\mathbb{N}^k$-graded ring R, then any $F \in R_{\underline{j}}$ can be written as $F = \sum_{i=1}^{N(\underline{j})} c_i m_i$ where $c_i \in \mathbf{k}$. Suppose that $P \in \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$. For $F \in R_{\underline{j}}$ to vanish at P we require $F(P) = \sum_{i=1}^{N(\underline{j})} c_i m_i(P) = 0$. By considering the c_i's as unknowns, this equation gives us one linear condition. If $\mathbb{X} = \{P_1, \ldots, P_s\}$, then for $F \in R_{\underline{j}}$ to vanish on $\mathbb{X}$ we require that $F(P_1) = \cdots = F(P_s) = 0$. We then have a linear system of equations

$$\begin{bmatrix} m_1(P_1) & \cdots & m_{N(\underline{j})}(P_1) \\ \vdots & & \vdots \\ m_1(P_s) & \cdots & m_{N(\underline{j})}(P_s) \end{bmatrix} \begin{bmatrix} c_1 \\ \vdots \\ c_{N(\underline{j})} \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}.$$

The number of linearly independent solutions is the rank of the matrix on the left. For a general enough set of points, we expect this rank to be as large as possible. By [16, Proposition 4.3] the rank of this matrix equals $H_{\mathbb{X}}(\underline{j})$, so we expect a general enough set of s points $\mathbb{X} \subseteq \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$ to have the Hilbert function $H_{\mathbb{X}}(\underline{j}) = \min\{N(\underline{j}), s\}$ for all $\underline{j} \in \mathbb{N}^k$. Proceeding analogously as in the case of points in $\mathbb{P}^n$, we make the following definition.

Definition 3.1. Let $\mathbb{X}$ be a finite set of s points in $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$ with Hilbert function $H_{\mathbb{X}}$. If

$$H_{\mathbb{X}}(\underline{j}) = \min\{N(\underline{j}), s\} \qquad \text{for all } \underline{j} \in \mathbb{N}^k,$$

then the Hilbert function is called *maximal.* A set of s points is said to be in *generic position* if its Hilbert function is maximal.

We now show the existence of sets of points in generic position by demonstrating that "most" sets of s points in $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$ are in generic position. We shall denote $(\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}) \times \cdots \times (\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k})$ (s times) by $(\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k})^s$.

Theorem 3.2. *The s-tuples of points of $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$, $(P_1, \ldots, P_s)$, considered as points of $(\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k})^s$, which are in generic position form a non-empty open subset of $(\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k})^s$.*

Proof. Since the case $k = 1$ is found in [6], we can assume that $k \geq 2$. Let $\{m_1, \ldots, m_{N(\underline{j})}\}$ be the $N(\underline{j})$ monomials of degree $\underline{j} = (j_1, \ldots, j_k) \in \mathbb{N}^k$ in R. By composing the product of j_i-uple embeddings with the Segre embedding we have a morphism $\nu_{\underline{j}} : \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k} \to \mathbb{P}^{N(\underline{j})-1}$ defined by

$$P = [a_{1,0} : \cdots : a_{1,n_1}] \times \cdots \times [a_{k,0} : \cdots : a_{k,n_k}] \longmapsto [m_1(P) : \cdots : m_{N(\underline{j})}(P)],$$

i.e., $m_i(P)$ is the monomial m_i evaluated at P. This induces a morphism

$$\varphi_{\underline{j}} = (\nu_{\underline{j}} : \cdots : \nu_{\underline{j}}) : (\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k})^s \longrightarrow \left(\mathbb{P}^{N(\underline{j})-1}\right)^s = V_{\underline{j}}.$$

By [6] there exists a nonempty open subset $W_{\underline{j}} \subseteq V_{\underline{j}}$ with the property that each point $(Q_1, \ldots, Q_s) \in W_{\underline{j}}$ corresponds to a set of s points in $\mathbb{P}^{N(\underline{j})-1}$ in generic position. In particular, each point of $W_{\underline{j}}$ corresponds to a set of s points in $\mathbb{P}^{N(\underline{j})-1}$ that impose $\min\{s, N(\underline{j})\}$ conditions on linear forms.

Because $\nu_{\underline{j}}$ does not vanish everywhere, $U_{\underline{j}} := \varphi^{-1}(W_{\underline{j}})$ is a non-empty open subset of $(\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k})^s$. Furthermore, because $\nu_{\underline{j}}$ induces an isomorphism between the linear forms of $\mathbb{P}^{N(\underline{j})-1}$ and the forms of $R_{\underline{j}}$, if $(P_1, \ldots, P_s) \in U_{\underline{j}}$, then as a set of points $\mathbb{X} = \{P_1, \ldots, P_s\}$ of $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$ we have $H_{\mathbb{X}}(\underline{j}) = \min\{s, N(\underline{j})\}$. Hence $U = \bigcap_{\underline{j} \in \mathbb{N}^k} U_{\underline{j}}$ consists of those s-tuples which correspond to sets of s points in $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$ with maximal Hilbert functions.

To complete the proof, it suffices to show that the intersection $U = \bigcap_{\underline{j} \in \mathbb{N}^k} U_{\underline{j}}$ can be taken to be finite, and thus U is open. Suppose $\underline{j} \in \mathbb{N}^k$ is such that $s = \min\{s, N(\underline{j})\}$. We claim that $U_{\underline{j}} \subseteq U_{\underline{j}'}$ for all $\underline{j} \leq \underline{j}'$. Indeed, take $(P_1, \ldots, P_s) \in U_{\underline{j}}$. So, if $\mathbb{X} = \{P_1, \ldots, P_s\}$, we have $H_{\mathbb{X}}(\underline{j}) = s$. Since the Hilbert function strictly increases until it stabilizes and is bounded by s, $H_{\mathbb{X}}(\underline{j}') = s < N(\underline{j}')$ for all $\underline{j} \leq \underline{j}'$. Thus $(P_1, \ldots, P_s) \in U_{\underline{j}'}$ as desired. Setting $D_1 = \{\underline{j} \in \mathbb{N}^k \mid N(\underline{j}) < s\}$ and $D_2 = \min\{\underline{j} \in \mathbb{N}^k \mid N(\underline{j}) \geq s\}$, we thus have $D = D_1 \cup D_2$ is a finite set and $U = \bigcap_{\underline{j} \in D} U_{\underline{j}}$. □

For any finite set of points $\mathbb{X} \subseteq \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$, we have $\text{K-}\dim R/I_{\mathbb{X}} = k$. However, it was shown in [17, Proposition 2.6] that the depth of $R/I_{\mathbb{X}}$ may take on any value in $\{1, \ldots, k\}$. When $\mathbb{X}$ is in generic position the depth can be calculated. We begin with a lemma.

Lemma 3.3. *Let $n, l \geq 1$ be integers. Then $\binom{n+l+1}{l+1} \leq \binom{n+l}{l}(n+1)$.*

Proof. Note that $\binom{n+l+1}{l+1} = \binom{n+l}{l} \cdot \frac{(n+l+1)}{(l+1)} = \binom{n+l}{l}\left(1 + \frac{n}{l+1}\right)$. The inequality now follows since $l \geq 1$, and thus $(1 + \frac{n}{l+1}) \leq (1 + n)$. □

Theorem 3.4. *If $\mathbb{X}$ is a set of $s > 1$ points in generic position in $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$, then* $\operatorname{depth} R/I_{\mathbb{X}} = 1$. *In particular, $R/I_{\mathbb{X}}$ is Cohen-Macaulay if and only if $k = 1$.*

Proof. By Lemma 1.1 $\operatorname{depth} R/I_{\mathbb{X}} \geq 1$. We show that equality holds. Without loss of generality, take $n_1 \leq n_2 \leq \cdots \leq n_k$ and let l be the minimal integer such that $\binom{n_1+l}{l} \geq s$. Since $\mathbb{X}$ is in generic position, $H_{\mathbb{X}}(l, 0, \ldots, 0) = \min\left\{\binom{n_1+l}{l}, s\right\} = s$.

Claim. If $\underline{j} \in \mathbb{N}^k$ and $\underline{j} > (l-1, 0, \ldots, 0)$, then $H_{\mathbb{X}}(\underline{j}) = s$.

Proof of the Claim. If $j_1 > l-1$, then $\binom{n_1+j_1}{j_1} \geq \binom{n_1+l}{l}$. Thus $N(\underline{j}) \geq \binom{n_1+l}{l} \geq \min\left\{\binom{n_1+l}{l}, s\right\} = s$, and hence, $H_{\mathbb{X}}(\underline{j}) = s$. So, suppose $j_1 = l-1$. Since $\underline{j} > (l-1, 0, \ldots, 0)$, there exists $m \in \{2, \ldots, k\}$ such that $j_m > 0$. Since $n_1 \leq n_m$, we have the following inequalities: $N(\underline{j}) \geq \binom{n_1+l-1}{l-1}\binom{n_m+j_m}{j_m} \geq \binom{n_1+l-1}{l-1}\binom{n_1+1}{1}$. By Lemma 3.3, we also have $\binom{n_1+l-1}{l-1}(n_1+1) \geq \binom{n_1+l}{l}$. Hence, $N(\underline{j}) \geq \binom{n_1+l}{l} \geq \min\left\{\binom{n_1+l}{l}, s\right\} = s$. Therefore, $H_{\mathbb{X}}(\underline{j}) = s$, as desired. □

Since $\overline{x}_{1,0}$ is a non-zero divisor of $R/I_{\mathbb{X}}$ we have the short exact sequence

$$0 \longrightarrow (R/I_{\mathbb{X}})\,(-e_1) \stackrel{\times \overline{x}_{1,0}}{\longrightarrow} R/I_{\mathbb{X}} \longrightarrow R/(I_{\mathbb{X}}, x_{1,0}) = R/J \longrightarrow 0$$

where $J = (I_{\mathbb{X}}, x_{1,0})$. Thus the Hilbert function of R/J is $H_{R/J}(\underline{j}) = H_{\mathbb{X}}(\underline{j}) - H_{\mathbb{X}}(\underline{j} - e_1)$ for all $\underline{j} \in \mathbb{N}^k$, where $H_{\mathbb{X}}(\underline{j}) = 0$ if $\underline{j} \not\geq \underline{0}$. From the claim, it follows that if $\underline{j} > (l, 0, \ldots, 0)$, then

$$H_{R/J}(\underline{j}) = H_{\mathbb{X}}(\underline{j}) - H_{\mathbb{X}}(\underline{j} - e_1) = s - s = 0.$$

On the other hand, if $\underline{j} = (l, 0, \ldots, 0)$, then

$$H_{R/J}(\underline{j}) = H_{\mathbb{X}}(le_1) - H_{\mathbb{X}}((l-1)e_1) = s - \binom{n_1 + l - 1}{l - 1} > 0.$$

Since $s \neq 1$ there exists a non-constant element $F \in R_{le_1}$ such that $0 \neq \overline{F} \in R/J$.

It suffices to demonstrate that all the non-constant homogeneous elements of R/J are annihilated by $\overline{F}$, and hence, $\operatorname{depth} R/J = 0$. So, suppose that $G \in R$ is such that $0 \neq \overline{G} \in R/J$. Without loss of generality we can take G to be an $\mathbb{N}^k$-homogeneous element with $\deg G = (j_1, \ldots, j_k) > \underline{0}$. Now $\deg FG = (j_1 + l, j_2, \ldots, j_k) > (l, 0, \ldots, 0)$. Since $H_{R/J}(j_1 + l, j_2, \ldots, j_k) = 0$, it follows that $FG \in J$. $\square$

Remark 3.5. If $s = 1$, then $\operatorname{depth} R/I_{\mathbb{X}} = k$ because the ideal of a point is a complete intersection.

We now apply Theorem 2.2 to describe the degrees of the generators of $I_{\mathbb{X}}$ when $\mathbb{X}$ is in generic position. We introduce some notation: if $E = \{\underline{j}_1, \ldots, \underline{j}_l\} \subseteq \mathbb{N}^k$, and $\underline{i} \in \mathbb{N}^k$, then

$$E + \underline{i} := \left\{\underline{j}_1 + \underline{i}, \underline{j}_2 + \underline{i}, \ldots, \underline{j}_l + \underline{i}\right\} \subseteq \mathbb{N}^k.$$

Also, let $\mathcal{D} := \min\{\underline{i} \in \mathbb{N}^k \mid (I_{\mathbb{X}})_{\underline{i}} \neq 0\} = \min\{\underline{i} \in \mathbb{N}^k \mid H_{\mathbb{X}}(\underline{i}) < N(\underline{i})\}$. (Recall that $\min \mathcal{S}$ with $\mathcal{S} \subseteq \mathbb{N}^k$ is the set of minimal elements of $\mathcal{S}$ with respect to the partial ordering $\underline{i} \geq \underline{j}$ if $i_h \geq j_j$ for all h.) When $\mathbb{X}$ is a set of s points in generic position we have $\mathcal{D} = \min\{\underline{i} \in \mathbb{N}^k \mid s < N(\underline{i})\}$.

Theorem 3.6. *Let $I_{\mathbb{X}}$ be the defining ideal of a set of s points in $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$ in generic position, and set $\mathcal{T} = \mathcal{D} \cup \left(\bigcup_{i=1}^k \mathcal{D} + e_i\right)$. If f is a generator of $I_{\mathbb{X}}$, then $\deg f \in \mathcal{T}$. In particular, $I_{\mathbb{X}} = \left\langle \bigoplus_{\underline{i} \in \mathcal{T}} (I_{\mathbb{X}})_{\underline{i}} \right\rangle$.*

Proof. The proof consists of two parts.

Step 1. We use Theorem 2.2 to show that $I_{\mathbb{X}} = \left\langle \bigoplus_{\underline{i} \in \mathcal{T}'} (I_{\mathbb{X}})_{\underline{i}} \right\rangle$ where

$$\mathcal{T}' = \bigcup_{\{l_1, \ldots, l_t\} \in \mathcal{P}([k])} \mathcal{D} + (e_{l_1} + \cdots + e_{l_t})$$

and $\mathcal{P}([k])$ denotes the power set of $[k]$. It is enough to show that $\mathcal{E} \subseteq \mathcal{T}'$. To do this, we need to first show that $\mathcal{D} \subseteq \mathcal{E}$. If $\underline{i} \in \mathcal{D}$, then $H_{\mathbb{X}}(\underline{i} - 2e_l) < H_{\mathbb{X}}(\underline{i} - e_l) \leq H_{\mathbb{X}}(\underline{i}) = s$ for all $l \in [k]$. So $\underline{i} \in \mathcal{D} \subseteq \mathcal{B}$, but $\underline{i} \notin \mathcal{A}$, hence $\underline{i} \in \mathcal{E}$.

Suppose $\underline{j} \in \mathcal{E}$. Then there exists $\underline{i} \in \mathcal{D} \subseteq \mathcal{E}$ such that $\underline{j} \geq \underline{i}$. We can thus write $\underline{j} = (i_1 + m_1, \ldots, i_k + m_k)$ where $\underline{i} = (i_1, \ldots, i_k)$. If $m_1 = \cdots = m_k = 0$, then $\underline{j} = \underline{i}$ and hence $\underline{j} \in \mathcal{E}$. So, suppose $m_l \geq 1$ for some $l \in [k]$. If $m_l \geq 2$, then $H_{\mathbb{X}}(\underline{j}) = H_{\mathbb{X}}(\underline{j} - e_l) = H_{\mathbb{X}}(\underline{j} - 2e_l) = s$ since $\underline{j} - 2e_l \geq \underline{i}$. But then $\underline{j} \in \mathcal{A}$, so $\underline{j} \notin \mathcal{E}$. Hence, for each $l \in [k]$, $m_l = 0$ or 1. So, if $m_{l_1} = \cdots = m_{l_t} = 1$, and 0 otherwise, then $\underline{j} = \underline{i} + (e_{l_1} + \cdots + e_{l_t}) \in \mathcal{T}'$.

Step 2. If F is a generator of $I_{\mathbb{X}}$ with $\deg F = \underline{j}$, then the previous step implies there exists $\underline{i} \in \mathcal{D}$ such that $\underline{j} = \underline{i} + e_{l_1} + \cdots + e_{l_t}$ for some subset $\{l_1, \ldots, l_t\} \subseteq [k]$. We wish to show that $t = 0$ or 1, i.e., $\deg F = \underline{i}$, or $\deg F = \underline{i} + e_l$ from some $l \in [k]$.

Let $\underline{i} \in \mathcal{D}$ and let $\{l_1, \ldots, l_t\}$ be any subset of $[k]$ with $t \geq 2$. Set $\underline{j} = \underline{i} + e_{l_1} + \cdots + e_{l_t}$. If we can show that $(I_{\mathbb{X}})_{\underline{j}} = R_{e_{l_1}}(I_{\mathbb{X}})_{\underline{j} - e_{l_1}} + R_{e_{l_2}}(I_{\mathbb{X}})_{\underline{j} - e_{l_2}}$ then we shall be finished because this implies that $(I_{\mathbb{X}})_{\underline{j}}$ contains no new generators.

By Remark 1.2, $\overline{x}_{l_2,0}$ is a non-zero divisor. Set $S = \mathbf{k}[x_{1,0},\ldots,\widehat{x}_{l_2,0},\ldots,x_{k,n_k}] \cong R/(x_{l_2,0})$. For each $\underline{t} \in \mathbb{N}^k$ we have the short exact sequence of vector spaces:

$$0 \longrightarrow (I_{\mathbb{X}})_{\underline{t}-e_{l_2}} \stackrel{\times x_{l_2,0}}{\longrightarrow} (I_{\mathbb{X}})_{\underline{t}} \longrightarrow (I_{\mathbb{X}}/x_{l_2,0}I_{\mathbb{X}})_{\underline{t}} \cong ((I_{\mathbb{X}},x_{l_2,0})/(x_{l_2,0}))_{\underline{t}} \longrightarrow 0.$$

This gives $\dim_{\mathbf{k}}(I_{\mathbb{X}})_{\underline{t}} = \dim_{\mathbf{k}}(I_{\mathbb{X}})_{\underline{t}-e_{l_2}} + \dim_{\mathbf{k}}((I_{\mathbb{X}},x_{l_2,0})/(x_{l_2,0}))_{\underline{t}}$.

Since $\mathbb{X}$ is in generic position, $H_{\mathbb{X}}(\underline{j}) = H_{\mathbb{X}}(\underline{j}-e_{l_2}) = H_{\mathbb{X}}(\underline{j}-e_{l_1}) = H_{\mathbb{X}}(\underline{j}-e_{l_1}-e_{l_2}) = s$. Thus, we can use the short exact sequence

$$0 \longrightarrow R/I_{\mathbb{X}}(-e_{l_2}) \stackrel{\times \overline{x}_{l_2,0}}{\longrightarrow} R/I_{\mathbb{X}} \longrightarrow R/(I_{\mathbb{X}},x_{l_2,0}) \cong \frac{R/(x_{l_2,0})}{(I_{\mathbb{X}},x_{l_2,0})/(x_{l_2,0})} \longrightarrow 0$$

to show that $((I_{\mathbb{X}},x_{l_2,0})/(x_{l_2,0}))_{\underline{j}-e_{l_1}} \cong S_{\underline{j}-e_{l_1}}$ and $((I_{\mathbb{X}},x_{l_2,0})/(x_{l_2,0}))_{\underline{j}} \cong S_{\underline{j}}$.

Set $W = R_{e_{l_1}}(I_{\mathbb{X}})_{\underline{j}-e_{l_1}} + R_{e_{l_2}}(I_{\mathbb{X}})_{\underline{j}-e_{l_2}}$. So $W \subseteq (I_{\mathbb{X}})_{\underline{j}}$. Because $x_{l_2,0}$ is a non-zero divisor

$$\dim_{\mathbf{k}} W = \dim_{\mathbf{k}}(I_{\mathbb{X}})_{\underline{j}-e_{l_2}} + \dim_{\mathbf{k}} W' = \dim_{\mathbf{k}}(I_{\mathbb{X}})_{\underline{j}} - \dim_{\mathbf{k}} S_{\underline{j}} + \dim_{\mathbf{k}} W'$$

where $W' = \{f'' \mid f = f'x_{l_2,0} + f'',\ f \in W\}$. The vector space W' can be viewed as a subspace of $S_{\underline{j}}$. It now suffices to show that $S_{\underline{j}} \subseteq W'$ because then $\dim_{\mathbf{k}} W = \dim_{\mathbf{k}}(I_{\mathbb{X}})_{\underline{j}}$, and thus, $W = (I_{\mathbb{X}})_{\underline{j}}$. So, let $f \in S_{\underline{j}}$. Then $f = f_0x_{l_1,0} + \cdots + f_{n_{l_1}}x_{l_1,n_{l_1}}$ with $f_i \in S_{\underline{j}-e_{l_1}}$. Since $S_{\underline{j}-e_{l_1}} \cong ((I_{\mathbb{X}},x_{l_2,0})/(x_{l_2,0}))_{\underline{j}-e_{l_1}}$, there exists (with a slight abuse of notation) $F_i \in (I_{\mathbb{X}})_{\underline{j}-e_{l_1}}$ such that $F_i = g_ix_{l_2,0} + f_i$. But then $F_0x_{l_1,0} + \cdots + F_{n_{l_1}}x_{l_1,n_{l_1}} \in W$, and hence, $f \in W'$. □

Remark 3.7. If $k = 1$ and $\mathbb{X}$ is a set of s points in generic position, then we obtain the well known result that $I_{\mathbb{X}} = \langle I_d \oplus I_{d+1}\rangle$ where $d = \min\{i \mid \binom{n+i}{i} > s\}$. [9, Lemma 4.2] is a proof of this theorem in the special case that $\mathbb{X}$ is a set of points in generic position in $\mathbb{P}^1 \times \mathbb{P}^1$. If $\mathbb{X}$ is a set of points in generic position in $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$, and if we set $d_i := \min\left\{d \;\middle|\; \binom{n_i+d}{d} \geq |\mathbb{X}|\right\}$ and $D := \max\{d_1+1,\ldots,d_k+1\}$, then the above result implies that $I_{\mathbb{X}}$, considered as an $\mathbb{N}^1$-graded ideal of R, is generated by forms of degree $\leq D$. This is extended in [13] to show that $\operatorname{reg}(I_{\mathbb{X}}) = D$, where $\operatorname{reg}(I_{\mathbb{X}})$ is the Castelnuovo-Mumford regularity of $I_{\mathbb{X}}$.

Corollary 3.8. *Let $\mathbb{X}$ be a set of s points in generic position in $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$ with Hilbert function $H_{\mathbb{X}}$. If there exists $l,m \in [k]$ (not necessarily distinct) and $\underline{j} \in \mathbb{N}^k$ such that $H_{\mathbb{X}}(\underline{j}) = H_{\mathbb{X}}(\underline{j}-e_l) = H_{\mathbb{X}}(\underline{j}-e_l-e_m) = s$, then $(I_{\mathbb{X}})_{\underline{j}}$ contains no generators of $I_{\mathbb{X}}$.*

Proof. If $l = m$, then this is simply Theorem 2.1. If $l \neq m$, then $H_{\mathbb{X}}(\underline{j}-e_m) = s$ because $\underline{j}-e_l-e_m \leq \underline{j}-e_m$ and $\mathbb{X}$ is in generic position. Arguing as in Step 2 of Theorem 3.6 we have $(I_{\mathbb{X}})_{\underline{j}} = R_{e_l}(I_{\mathbb{X}})_{\underline{j}-e_l} + R_{e_m}(I_{\mathbb{X}})_{\underline{j}-e_m}$. □

4. On the value of $\nu(I_{\mathbb{X}})$ for points in generic position

In this section we study $\nu(I_{\mathbb{X}})$, the minimal number of generators of $I_{\mathbb{X}}$, when $\mathbb{X}$ is a set of points in generic position. Unless specified otherwise, the set of points under consideration will be non-degenerate, that is, $|\mathbb{X}| > \max\{n_1,\ldots,n_k\}$. We give an upper bound on $\nu(I_{\mathbb{X}})$ that can be calculated from $n_1,\ldots,n_k$ and $|\mathbb{X}| = s$. We also show that calculating $\nu(I_{\mathbb{X}})$ is equivalent to calculating the dimensions of specific vector spaces. In some special cases, we are able to compute these dimensions.

So, suppose $\mathbb{X}$ is a non-degenerate set of s points in generic position in $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$. By Theorem 3.6, if $\underline{i} \in \mathcal{D} = \min\{\underline{i} \in \mathbb{N}^k \mid s < N(\underline{i})\} = \min\{\underline{i} \mid (I_{\mathbb{X}})_{\underline{i}} \neq 0\}$, by degree considerations $(I_{\mathbb{X}})_{\underline{i}}$ cannot be generated by elements of smaller degree. So the $\dim_{\mathbf{k}}(I_{\mathbb{X}})_{\underline{i}} = N(\underline{i}) - H_{\mathbb{X}}(\underline{i})$ linearly independent elements of $(I_{\mathbb{X}})_{\underline{i}}$ must be generators of $I_{\mathbb{X}}$. This gives a crude bound on $\nu(I_{\mathbb{X}})$:

$$\nu(I_{\mathbb{X}}) \geq \sum_{\underline{i}\in\mathcal{D}} \dim_{\mathbf{k}}(I_{\mathbb{X}})_{\underline{i}} = \sum_{\underline{i}\in\mathcal{D}} (N(\underline{i}) - s).$$

By Theorem 3.6, to compute $\nu(I_{\mathbb{X}})$ it suffices to calculate the number of generators of $I_{\mathbb{X}}$ in $(I_{\mathbb{X}})_{\underline{j}}$ for each $\underline{j} \in \bigcup_{l=1}^{k}(\mathcal{D}+e_l)$.

We wish to describe a subset of $\bigcup_{l=1}^{k}(\mathcal{D}+e_l)$ such that for each $\underline{j}$ in this subset, $(I_{\mathbb{X}})_{\underline{j}}$ contains no new generators of $I_{\mathbb{X}}$. We introduce some suitable notation. For each $\underline{i} \in \mathcal{D}$ set

$$\mathbb{D}_{\underline{i}} := \{\underline{j} \in \mathbb{N}^k \mid \underline{j} \geq \underline{i}\} \backslash \{\underline{i}, \underline{i}+e_1, \underline{i}+e_2, \ldots, \underline{i}+e_k\}.$$

If follows that $\underline{j} \in \mathbb{D}_{\underline{i}}$ if and only if $\underline{j} - e_{l_1} - e_{l_2} \geq \underline{i}$ for some not necessarily distinct $l_1, l_2 \in [k]$.

Lemma 4.1. *Let $\mathbb{X}$ be a non-degenerate set of s points in $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$. With the notation as above, suppose $\underline{j} \in \left[\bigcup_{l=1}^{k}(\mathcal{D}+e_l)\right] \cap \left[\bigcup_{\underline{i}\in\mathcal{D}} \mathbb{D}_{\underline{i}}\right]$. Then $I_{\mathbb{X}}$ has no generator of degree $\underline{j}$.*

Proof. Since $\underline{j} \in \mathbb{D}_{\underline{i}}$ for some $\underline{i} \in \mathcal{D}$, $\underline{j} - e_{l_1} - e_{l_2} \geq \underline{i}$ for some not necessarily distinct $l_1, l_2 \in [k]$. Because $\mathbb{X}$ is in generic position, we have $H_{\mathbb{X}}(\underline{j} - e_{l_1} - e_{l_2}) = H_{\mathbb{X}}(\underline{j} - e_{l_1}) = H_{\mathbb{X}}(\underline{j}) = s$, and so the conclusion follows from Corollary 3.8. □

Set

$$\mathbb{D} := \left[\bigcup_{l=1}^{k}(\mathcal{D}+e_l)\right] \setminus \left[\bigcup_{\underline{i}\in\mathcal{D}} \mathbb{D}_{\underline{i}}\right].$$

Because of Lemma 4.1, to determine $\nu(I_{\mathbb{X}})$ it is enough to count the number of generators of $I_{\mathbb{X}}$ with degree $\underline{j} \in \mathbb{D}$.

So, let $\underline{j} \in \mathbb{D}$. Since $\underline{j} \in \bigcup_{l=1}^{k}(\mathcal{D}+e_l)$, we can associate to $\underline{j}$ a unique subset $L_{\underline{j}} := \{l_1, \ldots, l_t\} \subseteq [k]$ such that $\underline{j} \in \mathcal{D}+e_{l_m}$ for each $l_m \in L_{\underline{j}}$ but $\underline{j} \notin \mathcal{D}+e_l$ if $l \in [k] \backslash L_{\underline{j}}$. For each $l_m \in L_{\underline{j}}$ there then exists a unique $\underline{i}_{l_m} \in \mathcal{D}$ such that $\underline{j} = \underline{i}_{l_m} + e_{l_m}$. So, for each $l_m \in L_{\underline{j}}$ we can define $W_{l_m, \underline{i}_{l_m}} := R_{e_{l_m}}(I_{\mathbb{X}})_{\underline{i}_{l_m}} \subseteq (I_{\mathbb{X}})_{\underline{j}}$. For each $\underline{j} \in \mathbb{D}$ we set

$$W_{\underline{j}} := W_{l_1,\underline{i}_{l_1}} + \cdots + W_{l_t,\underline{i}_{l_t}} = \sum_{l_m \in L_{\underline{j}}} W_{l_m,\underline{i}_{l_m}} \subseteq (I_{\mathbb{X}})_{\underline{j}}.$$

Thus $W_{\underline{j}}$ is the subvector space of $(I_{\mathbb{X}})_{\underline{j}}$ that consists of all the forms in $I_{\mathbb{X}}$ of degree $\underline{j}$ that come from forms of lower degree in $I_{\mathbb{X}}$. The number of new generators of $I_{\mathbb{X}}$ of degree $\underline{j}$ with $\underline{j} \in \mathbb{D}$ is then

$$\dim_{\mathbf{k}}(I_{\mathbb{X}})_{\underline{j}} - \dim_{\mathbf{k}}(W_{l_1,\underline{i}_{l_1}} + \cdots + W_{l_t,\underline{i}_{l_t}}) = N(\underline{j}) - s - \dim_{\mathbf{k}} W_{\underline{j}}.$$

We summarize this discussion with the following theorem.

Theorem 4.2. *Let $\mathbb{X}$ be a non-degenerate set of s points in generic position in $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$. With the notation as above*

$$\begin{aligned}\nu(I_{\mathbb{X}}) &= \sum_{\underline{i}\in\mathcal{D}} \dim_{\mathbf{k}}(I_{\mathbb{X}})_{\underline{i}} + \sum_{\underline{j}\in\mathbb{D}} \left(\dim_{\mathbf{k}}(I_{\mathbb{X}})_{\underline{j}} - \dim_{\mathbf{k}} W_{\underline{j}}\right) \\ &= \sum_{\underline{i}\in\mathcal{D}} (N(\underline{i}) - s) + \sum_{\underline{j}\in\mathbb{D}} \left(N(\underline{j}) - s - \dim_{\mathbf{k}} W_{\underline{j}}\right).\end{aligned}$$

Computing $\nu(I_{\mathbb{X}})$ is thus equivalent to computing $\dim_{\mathbf{k}} W_{\underline{j}}$ for each $\underline{j} \in \mathbb{D}$. Arguing as in [7, Proposition 7] one has the following lower bounds:

Lemma 4.3. *Suppose $\underline{j} \in \mathbb{D}$ and $L_{\underline{j}} = \{l_1, \ldots, l_t\}$. For every $l_m \in L_{\underline{j}}$*

$$\dim_{\mathbf{k}} W_{\underline{j}} \geq \dim_{\mathbf{k}} W_{l_m}, \underline{i}_{l_m} \geq 2 \dim_{\mathbf{k}}(I_{\mathbb{X}})_{\underline{i}_{l_m}}.$$

Combining Lemma 4.3 with Theorem 4.2 gives us an upper bound on $\nu(I_{\mathbb{X}})$.

Corollary 4.4. *Let $\mathbb{X}$ be a non-degenerate set of s points in generic position in $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$. With the notation as above*

$$\nu(I_{\mathbb{X}}) \quad \leq \quad \sum_{\underline{i} \in \mathcal{D}} (N(\underline{i}) - s) + \sum_{\underline{j} \in \mathbb{D}} \big(N(\underline{j}) - 2N(\underline{j} - e_{l_1}) + s\big)$$

where $l_1 \in L_{\underline{j}} = \{l_1, \ldots, l_t\}$ for $\underline{j} \in \mathbb{D}$.

Proof. For each $\underline{j} \in \mathbb{D}$, let $l_1 \in L_{\underline{j}}$. Then $\dim_{\mathbf{k}} W_{\underline{j}} \geq 2 \dim_{\mathbf{k}} (I_{\mathbb{X}})_{\underline{j} - e_{l_1}} = 2N(\underline{j} - e_{l_1}) - 2s$. Now use Theorem 4.2. □

In general, calculating $\dim_{\mathbf{k}} W_{\underline{j}}$ for each $\underline{j} \in \mathbb{D}$ is a very difficult problem. Indeed, if $L_{\underline{j}} = \{l_1, \ldots, l_t\}$, then there is no *a priori* formula for calculating $\dim_{\mathbf{k}} W_{l_m, \underline{i}_{l_m}} = \dim_{\mathbf{k}} (R_{e_{l_m}} (I_{\mathbb{X}})_{\underline{i}_{l_m}})$ for each $l_m \in L_{\underline{j}}$. The problem is further complicated when $|L_{\underline{j}}| = t \geq 2$ because then we need to know how $W_{l_m, \underline{i}_{l_m}}$ and $W_{l_n, \underline{i}_{l_n}}$ intersect in $(I_{\mathbb{X}})_{\underline{j}}$ for each $l_n, l_m \in L_{\underline{j}}$.

However, under some extra hypotheses on either $s = |\mathbb{X}|$ or $n_1, \ldots, n_k$ we can be quite explicit about $\dim_{\mathbf{k}} W_{\underline{j}}$ for *some* $\underline{j} \in \mathbb{D}$. The remaining results of this section are of this vein.

Lemma 4.5. *Let $\mathbb{X} \subseteq \mathbb{P}^1 \times \cdots \times \mathbb{P}^1$ be any finite set of points. Then, for any $\underline{i} \in \mathbb{N}^k$ and $l \in [k]$,*

$$\dim_{\mathbf{k}} \big(R_{e_l} (I_{\mathbb{X}})_{\underline{i}}\big) = 2 \dim_{\mathbf{k}} (I_{\mathbb{X}})_{\underline{i}} - \dim_{\mathbf{k}} (I_{\mathbb{X}})_{\underline{i} - e_l}$$

where $\dim_{\mathbf{k}} (I_{\mathbb{X}})_{\underline{i} - e_l} = 0$ if $\underline{i} - e_l \not\geq 0$.

Proof. The proof for the case $\mathbb{X} \subseteq \mathbb{P}^1 \times \mathbb{P}^1$ given in [8, Lemma 2.3] can be extended to this case. □

Theorem 4.6. *Let $\mathbb{X} \subseteq \mathbb{P}^1 \times \cdots \times \mathbb{P}^1$ be a set of $s > 1$ points in generic position. With the notation as above, suppose $\underline{j} \in \mathbb{D}$ with $L_{\underline{j}} = \{l\}$. Then*

$$\dim_{\mathbf{k}} W_{\underline{j}} = \begin{cases} \dim_{\mathbf{k}} (I_{\mathbb{X}})_{\underline{j}} = N(\underline{j}) - s & \text{if } N(\underline{j} - 2e_l) = s. \\ 2 \dim_{\mathbf{k}} (I_{\mathbb{X}})_{\underline{j} - e_l} = 2N(\underline{j} - e_l) - 2s & \text{if } N(\underline{j} - 2e_l) < s. \end{cases}$$

Proof. The hypothesis $L_{\underline{j}} = \{l\}$ implies $\underline{j} - e_l = \underline{i} \in \mathcal{D}$ but $\underline{j} - e_m \notin \mathcal{D}$ for any $m \in [k] \backslash \{l\}$. Hence $\dim_{\mathbf{k}} W_{\underline{j}} = \dim_{\mathbf{k}} (R_{e_l} (I_{\mathbb{X}})_{\underline{i}})$. Since $\underline{i} \in \mathcal{D}$, there does not exists an $\underline{i}' \in \mathcal{D}$ such that $\underline{i} - e_l \geq \underline{i}'$, and hence $\dim_{\mathbf{k}} (I_{\mathbb{X}})_{\underline{i} - e_l} = 0$. By Lemma 4.5 we thus have $\dim_{\mathbf{k}} W_{\underline{j}} = 2 \dim_{\mathbf{k}} (I_{\mathbb{X}})_{\underline{i}} = 2N(\underline{j} - e_l) - 2s$. The reader can now verify that $2N(\underline{j} - e_l) - 2s = N(\underline{j}) - s$ if $s = N(\underline{j} - 2e_l)$. □

Theorem 4.7. *Let $\mathbb{X}$ be a non-degenerate set of s points in generic position in $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$ and $\underline{j} \in \mathbb{D}$.*

(*i*) *If there exists $l \in [k]$ such that $N(\underline{j} - 2e_l) = s$, then $\dim_{\mathbf{k}} W_{\underline{j}} = \dim_{\mathbf{k}} (I_{\mathbb{X}})_{\underline{j}}$.*
(*ii*) *If $L_{\underline{j}} = \{l\}$ and $s = N(\underline{j} - e_l) - 1$, then $\dim_{\mathbf{k}} W_{\underline{j}} = n_l + 1$.*

Proof. (*i*) Since $N(\underline{j} - 2e_l) = s$, $H_{\mathbb{X}}(\underline{j} - 2e_l) = H_{\mathbb{X}}(\underline{j} - e_l) = s$. Now apply Theorem 2.1.

(*ii*) We are given that $\underline{j} - e_l \in \mathcal{D}$ and $\dim_{\mathbf{k}} (I_{\mathbb{X}})_{\underline{j} - e_l} = N(\underline{j} - e_l) - s = 1$. So let F be a basis for $(I_{\mathbb{X}})_{\underline{j} - e_l}$. Then $x_{l,0} F, \ldots, x_{l,n_l} F$ form a linearly independent basis of $W_{\underline{j}} = R_{e_l} (I_{\mathbb{X}})_{\underline{j} - e_l}$. □

5. On the expected value of $\nu(I_{\mathbb{X}})$

Let $\mathbb{X}$ be a non-degenerate set of points in generic position in $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$. In this section we are interested in determining if there is an expected value for $\nu(I_{\mathbb{X}})$. After showing that $\nu(I_{\mathbb{X}})$ is constant on some open subset of $(\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k})^s$, we give a lower bound for this value. When $k = 1$ the resulting lower bound is conjectured to equal $\nu(I_{\mathbb{X}})$ on some non-empty open subset of $(\mathbb{P}^n)^s$ by the Ideal Generation Conjecture. Therefore, it seems natural to expect that our generalized lower bound equals $\nu(I_{\mathbb{X}})$ on some non-empty open subset of $(\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k})^s$, thus generalizing the IGC to points in $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$. However, although we have found computationally that in many cases $\nu(I_{\mathbb{X}})$ equals the

lower bound, we show that there exist s and $n_1,\ldots,n_k$ for which $\nu(I_{\mathbb{X}})$ is always larger than this bound. We continue to use the notation of the previous sections.

If $(P_1,\ldots,P_s) \in (\mathbb{P}^{n_1}\times\cdots\times\mathbb{P}^{n_k})^s$ is such that $P_1,\ldots,P_s$ are distinct points, then we shall write $I(P_1,\ldots,P_s)$ to denote the defining ideal of $\{P_1,\ldots,P_s\} \subseteq \mathbb{P}^{n_1}\times\cdots\times\mathbb{P}^{n_k}$. Furthermore, if $\underline{j} \in \mathcal{D}$ with $L_{\underline{j}} = \{l_1,\ldots,l_t\}$, then we write $W(P_1,\ldots,P_s)_{\underline{j}}$ for the vector space $W(P_1,\ldots,P_s)_{\underline{j}} := R_{e_{l_1}} I_{\underline{j}-e_{l_1}} + \cdots + R_{e_{l_t}} I_{\underline{j}-e_{l_t}} \subseteq I_{\underline{j}}$ where $I = I(P_1,\ldots,P_s)$.

Theorem 5.1. *Let $s > \max\{n_1,\ldots,n_k\}$. Then there exists an open set $U \subseteq (\mathbb{P}^{n_1}\times\cdots\times\mathbb{P}^{n_k})^s$ such that if $(P_1,\ldots,P_s)\in U$, then $\dim_{\mathbf{k}} W(P_1,\ldots,P_s)_{\underline{j}}$ is the maximum possible for all $\underline{j}\in\mathbb{D}$. In particular, $\nu(I(P_1,\ldots,P_s))$ is constant for all $(P_1,\ldots,P_s)\in U$.*

Proof. It is enough to show that for each $\underline{j}\in\mathbb{D}$, there exists an open subset $U_{\underline{j}} \subseteq (\mathbb{P}^{n_1}\times\cdots\times\mathbb{P}^{n_k})^s$ with the property that if $(P_1,\ldots,P_s)\in U_{\underline{j}}$, then $\dim_{\mathbf{k}} W(P_1,\ldots,P_s)_{\underline{j}}$ is maximal. Then, since $|\mathbb{D}|<\infty$, the desired open set is $U = \bigcap_{\underline{j}\in\mathbb{D}} U_{\underline{j}}$.

So, let $\underline{j}\in\mathbb{D}$ and suppose $L_{\underline{j}} = \{l_1,\ldots,l_t\}$. For each $l_m \in L_{\underline{j}}$ set $\underline{i}_{l_m} := \underline{j} - e_{l_m}$. Let $W \subseteq (\mathbb{P}^{n_1}\times\cdots\times\mathbb{P}^{n_k})^s$ denote the open set of Theorem 3.2 consisting of the s distinct points in generic position. Then, using the proof of the claim found after Remark 2.8 in [4], we can show that there exists an open set $U_{l_m}\subseteq W$ such that for all $(P_1,\ldots,P_s)\in U_{l_m}$, $\dim_{\mathbf{k}} R_{e_{l_m}} I(P_1,\ldots,P_s)_{\underline{i}_{l_m}}$ is the maximum possible.

If we let $G_{l_m,1},\ldots,G_{l_m,N(\underline{i}_{l_m})-s}$ denote the $N(\underline{i}_{l_m})-s = \dim_{\mathbf{k}} I(P_1,\ldots,P_s)_{\underline{i}_{l_m}}$ distinct basis elements of $I(P_1,\ldots,P_s)_{\underline{i}_{l_m}}$, then the elements

$$\left\{x_{l_m,i}G_{l_m,j} \;\middle|\; 0\le i\le n_{l_m}, 1\le j\le N(\underline{i}_{l_m})-s\right\}$$

generate $R_{e_{l_m}} I(P_1,\ldots,P_s)_{\underline{i}_{l_m}}$. Set $M_{l_m} = (n_{l_m}+1)(N(\underline{i}_{l_m})-s)$ and form the $M_{l_m}\times N(\underline{j})$ matrix $\mathcal{M}_{l_m}$ which expresses how the $x_{l_m,i}G_{l_m,j}$'s are linear combinations of the $N(\underline{j})$ monomials of degree $\underline{j}$. Since $\operatorname{rank}\mathcal{M}_{l_m} = \dim_{\mathbf{k}} R_{e_{l_m}} I(P_1,\ldots,P_s)_{\underline{i}_{l_m}}$, this rank is maximal for all $(P_1,\ldots,P_s)\in U_{l_m}$.

Let $\mathcal{M}$ be the $\left(\sum_{l_m\in L_{\underline{j}}} M_{l_m}\right)\times N(\underline{j})$ matrix $\mathcal{M} := \begin{bmatrix}\mathcal{M}_{l_1}\\ \vdots \\ \mathcal{M}_{l_t}\end{bmatrix}$. Then the rank of $\mathcal{M}$ is equal to $\dim_{\mathbf{k}} W(P_1,\ldots,P_s)_{\underline{j}}$. The rank of $\mathcal{M}$ will therefore assume its maximal value on some open subset $U_{\underline{j}} \subseteq \bigcap_{l_m\in L_{\underline{j}}} U_{l_m}$. This is the desired set $U_{\underline{j}}$. $\square$

We can give a lower bound on $\nu(I_{\mathbb{X}})$ by using Theorem 4.2 and bounds on $\dim_{\mathbf{k}} W_{\underline{j}}$ for each $\underline{j}\in\mathbb{D}$. For each $l_m\in L_{\underline{j}}$ the dimension of $W_{l_m,\underline{i}_{l_m}}$ is bounded by

$$\dim_{\mathbf{k}} W_{l_m,\underline{i}_{l_m}} \le \min\left\{\dim_{\mathbf{k}}(I_{\mathbb{X}})_{\underline{j}},\ (n_{l_m}+1)\dim_{\mathbf{k}}(I_{\mathbb{X}})_{\underline{i}_{l_m}}\right\}. \tag{1}$$

Furthermore, if $l_m, l_n \in L_{\underline{j}}$ and $l_m \neq l_n$, then

$$\dim_{\mathbf{k}}(W_{l_m,\underline{i}_{l_m}} + W_{l_n,\underline{i}_{l_n}}) \le \dim_{\mathbf{k}} W_{l_m,\underline{i}_{l_m}} + \dim_{\mathbf{k}} W_{l_n,\underline{i}_{l_n}}.$$

We thus arrive at the following upper bound for $W_{\underline{j}}$:

$$\dim_{\mathbf{k}} W_{\underline{j}} \le \min\left\{\dim_{\mathbf{k}}(I_{\mathbb{X}})_{\underline{j}},\ \sum_{l_m\in L_{\underline{j}}}(n_{l_m}+1)\dim_{\mathbf{k}}(I_{\mathbb{X}})_{\underline{i}_{l_m}}\right\}. \tag{2}$$

Since the values of $\dim_{\mathbf{k}}(I_{\mathbb{X}})_{\underline{j}}$ and $\dim_{\mathbf{k}}(I_{\mathbb{X}})_{\underline{i}_{l_m}}$ are known because $\mathbb{X}$ is in generic position, combing the above upper bound with Proposition 4.2 results in the following lower bound:

Theorem 5.2. *Let $\mathbb{X}$ be a non-degenerate set of s points in generic position in $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$, and set*

$$v(s;n_1,\ldots,n_k) := \sum_{\underline{i}\in\mathcal{D}}(N(\underline{i})-s) + \sum_{\underline{j}\in\mathbb{D}} \max\left\{0,\ N(\underline{j}) - s - \sum_{l_m\in L_{\underline{j}}}(n_{l_m}+1)(N(\underline{i}_{l_m})-s)\right\}.$$

Then $\nu(I_{\mathbb{X}}) \geq v = v(s;n_1,\ldots,n_k)$.

Remark 5.3. If $k=1$, then

$$v = v(s;n) = \binom{d+n}{n} - s + \max\left\{0, \binom{d+1+n}{n} - s - (n+1)\left(\binom{d+n}{n} - s\right)\right\}$$

where $d = \min\left\{i \ \middle|\ \binom{n+i}{i} > s\right\}$. The Ideal Generation Conjecture conjectures that $\nu(I_{\mathbb{X}}) = v$ on some non-empty open subset of $(\mathbb{P}^n)^s$. Although known to be true in some cases (for $n=2$ see [4], for $n=3$ see [1], and for $s \gg n$ see [14]) the conjecture remains open in general. The conjecture was formulated using the heuristic argument that "generically" $\dim_{\mathbf{k}} W_{\underline{j}}$ should be as large as possible, thus implying equality in the bounds (1) and (2). It seems natural to extend this heuristic argument to points in $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$ to generalize the Ideal Generation Conjecture by expecting that $\nu(I_{\mathbb{X}}) = v(s;n_1,\ldots,n_k)$ for some non-empty open set of $(\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k})^s$. But as we show at the end of this section, sometimes $\nu(I_{\mathbb{X}}) > v$ if $k \geq 2$.

In [9, 10] Giuffrida, et al. computed the minimal free resolution of points in generic position in $\mathbb{P}^1 \times \mathbb{P}^1$, and in particular, showed that $\nu(I_{\mathbb{X}}) = v$ on some non-empty open set of $(\mathbb{P}^1 \times \mathbb{P}^1)^s$. Since their notation and approach is different than ours, for the convenience of the reader we make this connection more transparent.

Theorem 5.4. *There exists a non-empty open subset $U \subseteq (\mathbb{P}^1 \times \mathbb{P}^1)^s$ with $s \geq 2$ such that for all $(P_1,\ldots,P_s) \in U$, the points $\{P_1,\ldots,P_s\}$ are in generic position, and $\nu(I(P_1,\ldots,P_s)) = v(s;1,1)$.*

Proof. It suffices to show that the bound (2) for $\dim_{\mathbf{k}} W_{\underline{j}}$ is in fact an equality for each $\underline{j} \in \mathbb{D}$. If $\underline{j} = (i,j) \in \mathbb{D}$, there are two possibilities: $|L_{\underline{j}}| = 1$ or 2. In the former, by Theorem 4.6 we have equality in (2).

For the second case, by [10, Theorem 4.3] there exists a non-empty subset $U \subseteq (\mathbb{P}^1 \times \mathbb{P}^1)^s$ such that for all $\underline{j} = (i,j) \in \mathbb{D}$ with $|L_{\underline{j}}| = 2$ and for each $(P_1,\ldots,P_s) \in U$, we have

$$\dim_{\mathbf{k}}(I(P_1,\ldots,P_s))_{i,j} - \dim_{\mathbf{k}} W(P_1,\ldots,P_s)_{i,j} = \max\{0, -d_{i,j}\}.$$

Here, $d_{i,j}$ is the (i,j)th entry of what [10] call the second difference Hilbert matrix of $\mathbb{X} = \{P_1,\ldots,P_s\}$ which is computed from the Hilbert function on $\mathbb{X}$. Since $|L_{\underline{j}}| = 2$ and because $\mathbb{X}$ is in generic position, $H_{\mathbb{X}}$, written as a matrix, has the form

$$\begin{array}{r|cccc}
 & & {\scriptstyle (j-2)} & {\scriptstyle (j-1)} & {\scriptstyle j} \\
\hline
 & & \vdots & \vdots & \vdots \\
{\scriptstyle (i-2)} & \cdots & (i-1)(j-1) & (i-1)j & (i-1)(j+1) \\
{\scriptstyle (i-1)} & \cdots & i(j-1) & ij & s \\
{\scriptstyle i} & \cdots & (i+1)(j-1) & s & s
\end{array}$$

This local description of the Hilbert function, and the definition of $d_{i,j}$ on page 422 of [10] gives

$$\begin{aligned}
-d_{i,j} &= [(i+1)(j+1)-s] - [2((i+1)j-s) + 2((j+1)i-s)] \\
&= \dim_{\mathbf{k}}(I_{\mathbb{X}})_{i,j} - 2\dim_{\mathbf{k}}(I_{\mathbb{X}})_{i-1,j} - 2\dim_{\mathbf{k}}(I_{\mathbb{X}})_{i,j-1}.
\end{aligned}$$

We thus have equality in (2). □

We now show that $\nu(I_{\mathbb{X}})$ may not equal $v = v(s;n_1,\ldots,n_k)$ in general. We begin by showing that any example of points $\mathbb{X}$ with $\nu(I_{\mathbb{X}}) > v$ can be extended to an infinite family of examples.

Lemma 5.5. *Suppose that for every non-degenerate set $\mathbb{X}$ of s points in generic position in $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$ we have $\nu(I_{\mathbb{X}}) > v(s; n_1, \ldots, n_k)$. If $\mathbb{X}'$ is any non-degenerate set of s points in generic position in $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k} \times \mathbb{P}^{m_1} \times \cdots \times \mathbb{P}^{m_l}$, then $\nu(I_{\mathbb{X}'}) > v(s; n_1, \ldots, n_k, m_1, \ldots, m_l)$.*

Proof. Let $\mathbb{X}'$ be a set of s points in generic position in $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k} \times \mathbb{P}^{m_1} \times \cdots \times \mathbb{P}^{m_l}$. Let $I_{\mathbb{X}'}$ be the associated ideal and set $I := \bigoplus_{(i_1,\ldots,i_k)\in\mathbb{N}^k} (I_{\mathbb{X}'})_{(i_1,\ldots,i_k,0,\ldots,0)}$. Then I is isomorphic to an ideal $I_{\mathbb{X}} \subseteq \mathbf{k}[x_{1,0}, \ldots, x_{1,n_1}, \ldots, x_{k,0}, \ldots, x_{k,n_k}]$ where $I_{\mathbb{X}}$ is the defining ideal of a set $\mathbb{X}$ of s points in generic position in $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$.

By hypothesis, there exists $\underline{j} = (j_1, \ldots, j_k) \in \mathbb{N}^k$ such that $(I_{\mathbb{X}})_{\underline{j}}$ contains a generator that has not been accounted for by $v(s; n_1, \ldots, n_k)$. Hence $(I_{\mathbb{X}'})_{(j_1,\ldots,j_k,0,\ldots,0)} \cong (I_{\mathbb{X}})_{\underline{j}}$ contains a generator of $I_{\mathbb{X}'}$ that is not expected, and thus $\nu(I_{\mathbb{X}'})$ will be strictly larger than $v(s; n_1, \ldots, n_k, m_1, \ldots, m_l)$. $\square$

We now give a case where $\nu(I_{\mathbb{X}})$ fails to agree with the lower bound.

Theorem 5.6. *Let $\mathbb{X}$ be three points in generic position in $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$. Then $\nu(I_{\mathbb{X}}) > v(3; 1, 1, 1)$.*

Proof. It is enough to show the existence of some $\underline{j} \in \mathbb{D}$ for which we have a strict inequality in (2) for $\dim_{\mathbf{k}} W_{\underline{j}}$. Now $\{(1,1,0), (1,0,1), (0,1,1)\} \subseteq \mathcal{D} = \min\{\underline{i} \in \mathbb{N}^3 \mid N(\underline{i}) > 3\}$, and so $\underline{j} := (1,1,1) \in \mathbb{D}$ with $L_{\underline{j}} = \{1, 2, 3\}$. The expected dimension of $W_{\underline{j}}$ is

$$\min\left\{\dim_{\mathbf{k}}(I_{\mathbb{X}})_{\underline{j}}, \sum_{i\in\{1,2,3\}} 2\dim_{\mathbf{k}}(I_{\mathbb{X}})_{\underline{j}-e_i}\right\} = \min\left\{8-3, \sum_{i\in\{1,2,3\}} 2(4-3)\right\} = 5,$$

or equivalently, we expect $(I_{\mathbb{X}})_{\underline{j}}$ to contain no generators.

However, we claim $\dim_{\mathbf{k}} W_{\underline{j}} \leq 4$, and hence, $(I_{\mathbb{X}})_{\underline{j}}$ contains a new generator. Let P_1, P_2, P_3 be the distinct points of $\mathbb{X}$, and after a linear change of variables in each set of coordinates, we can assume $P_1 = [1:0] \times [1:0] \times [1:0]$, $P_2 = [1:a_1] \times [1:a_2] \times [1:a_3]$, and $P_3 = [1:b_1] \times [1:b_2] \times [1:b_3]$ with $a_i \neq b_i$ for $i = 1, 2, 3$. Because $\mathbb{X}$ is in generic position $\dim_{\mathbf{k}}(I_{\mathbb{X}})_{1,1,0} = \dim_{\mathbf{k}}(I_{\mathbb{X}})_{1,0,1} = \dim_{\mathbf{k}}(I_{\mathbb{X}})_{0,1,1} = 1$. To find a basis for each of these vector spaces, it suffices to find a form of the proper degree in $I_{\mathbb{X}}$. From our description of the points we can find such forms:

$$\begin{aligned} F_1 &:= (a_2b_1 - a_1b_2)x_1y_1 + a_2b_2(a_1 - b_1)x_1y_0 + a_1b_1(b_2 - a_2)x_0y_1 \in (I_{\mathbb{X}})_{1,1,0} \\ F_2 &:= (a_3b_1 - a_1b_3)x_1z_1 + a_3b_3(a_1 - b_1)x_1z_0 + a_1b_1(b_3 - a_3)x_0z_1 \in (I_{\mathbb{X}})_{1,0,1} \\ F_3 &:= (a_2b_3 - a_3b_2)y_1z_1 + a_3b_3(b_2 - a_2)y_1z_0 + a_2b_2(a_3 - b_3)y_0z_1 \in (I_{\mathbb{X}})_{0,1,1}. \end{aligned}$$

It follows that $z_0F_1, z_1F_1, y_0F_2, y_1F_2, x_0F_3, x_1F_3$ generate the vector space $W_{\underline{j}}$. A routine calculation will now verify that

$$\begin{aligned} a_1b_1(a_1 - b_1)x_0F_3 &= (a_1 - b_1)[a_3b_3z_0F_1 - a_2b_2y_0F_2] + (a_3b_1 - a_1b_3)z_1F_1 \\ &\quad -(a_2b_1 - b_2a_1)y_1F_2 \\ (a_1 - b_1)x_1F_3 &= (b_2 - a_2)y_1F_2 + (a_3 - b_3)z_1F_1. \end{aligned}$$

Thus, x_0F_3, x_1F_3 are in the vector space spanned by $z_0F_1, z_1F_1, y_0F_2, y_1F_2$, whence $\dim_{\mathbf{k}} W_{\underline{j}} \leq 4 < 5 =$ the expected dimension. $\square$

With this result we can construct examples with $\nu(I_{\mathbb{X}})$ arbitrarily larger than $v(s; n_1, \ldots, n_k)$.

Corollary 5.7. *Let $\mathbb{X}$ be three points in generic position in $\mathbb{P}^1 \times \cdots \times \mathbb{P}^1$ ($k \geq 3$ times). Then $\nu(I_{\mathbb{X}}) \geq v(3; 1, \ldots, 1) + \binom{k}{3}$.*

Proof. There are $\binom{k}{3}$ tuples $\underline{i} \in \mathbb{N}^k$ which have exactly three 1's and $k-3$ zeroes. Let $\underline{i}$ be such a tuple, and suppose that the three 1's are in i_1th, i_2th, and i_3th position. If $\pi_{i_1,i_2,i_3} : \mathbb{P}^1 \times \cdots \times \mathbb{P}^1 \to \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ is the projection map onto the i_1th, i_2th, and i_3th coordinates, then $\mathbb{Y} = \pi_{i_1,i_2,i_3}(\mathbb{X}) \subseteq \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ is a set of three points in generic position. Hence, $(I_{\mathbb{X}})_{\underline{i}} \cong (I_{\mathbb{Y}})_{1,1,1}$. But by Theorem 5.6 $(I_{\mathbb{Y}})_{1,1,1}$ contains

at least one generator not accounted for by $v(3;1,1,1)$, and thus, $(I_{\mathbb{X}})_{\underline{i}}$ has a generator not counted by $v(3;1,\ldots,1)$. □

Using CoCoA [3] we have computed $\nu(I_{\mathbb{X}})$ in the following ranges:

$$\begin{array}{lll} k=2 & 1 \leq n_1 \leq n_2 \leq 5 & n_2 < s \leq 20 \\ k=3 & 1 \leq n_1 \leq n_2 \leq n_3 \leq 5 & n_3 < s \leq 10 \\ k=4 & 1 \leq n_1 \leq n_2 \leq n_3 \leq n_4 \leq 5 & n_4 < s \leq 10. \end{array}$$

Besides the example of Theorem 5.6 (and those examples that are a consequence of Lemma 5.5) we found that

$$\nu(I_{\mathbb{X}}) > v(1+n+n;1+n+n) \text{ for } 1 \leq n \leq 7.$$

From this data it appears that $\nu(I_{\mathbb{X}}) > v(1+n+n;1,n,n)$ for all n. Notice that the example of Theorem 5.6 is also part of this family. Using CoCoA we found that in each of these cases $\dim_{\mathbf{k}} W_{1,1,1}$ is smaller than the expected dimension.

We point out, however, that in every other case the computed value of $\nu(I_{\mathbb{X}})$ agrees with $v(s;n_1,\ldots,n_k)$. These computations leads us to believe that $\nu(I_{\mathbb{X}}) = v(s;n_1,\ldots,n_k)$ for a large number s and $n_1,\ldots,n_k$. Moreover, we know of no counterexamples when $k \leq 2$. We conclude by giving some questions inspired by our computer examples.

Question 5.8. *For $s = (1+n+n)$ points in generic position in $\mathbb{P}^1 \times \mathbb{P}^n \times \mathbb{P}^n$ is $\nu(I_{\mathbb{X}})$ always the larger $v(s;1,n,n)$? Is this family of examples the only family where the lower bound fails to hold? If not, can we classify all s and $n_1,\ldots,n_k$ for which $\nu(I_{\mathbb{X}}) \neq v(s;n_1,\ldots,n_k)$? Does the lower bound value always hold in the case $k \leq 2$? How should a generalized Ideal Generalization Conjecture be formulated to account for these examples?*

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